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★**Dialgebras.**
Dialgebras and related operads, 7–66, *Lecture Notes in Math.*, 1763, *Springer*, *Berlin*, 2001.

The Leibniz homology and Leibniz algebras were introduced by the author in [*Cyclic homology*, Springer, Berlin, 1992; MR1217970]. Leibniz algebras are a non-anticommutative generalization of Lie algebras. A Leibniz algebra $\mathfrak{g}$ is a vector space over a field k, equipped with a bracket $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$, satisfying the identity $[[x,y],z]=[[x,z],y]+[x,[y,z]]$. The relationship between Lie and associative algebras is well known. One can ask whether there is a generalization of associative algebras which will play the same role for Leibniz algebras as associative algebras does for Lie algebras. The work under review gives a reasonable answer on this question. The author invents new type algebras—dialgebras. A dialgebra is a vector space equipped with two associative multiplications $\dashv,\vdash$ which satisfy the following axioms:

$$x\dashv(y\dashv z)=x\dashv(y\vdash z),$$

$$(x\vdash y)\dashv z=x\vdash(y\dashv z),$$

$$(x\dashv y)\vdash z=(x\vdash y)\vdash z.$$

If both multiplications coincide then one gets an associative algebra. Any dialgebra gives rise to a Leibniz algebra by $[x,y]=x\dashv y-y\vdash x$. In this way one gets a functor **Dias** $\to$ **Leib** from the category of dialgebras to the category of Leibniz algebras. It turns out that this functor has a left adjoint functor, which assigns the so-called universal enveloping dialgebra $\mathrm{Ud}(\mathfrak{g})$ to a Leibniz algebra $\mathfrak{g}$. For a dialgebra D one can consider an associative algebra D_{As} which is the quotient of D by the relation $x\dashv y=x\vdash y$. Then the Leibniz universal enveloping algebra $\mathrm{UL}(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to $\mathrm{Ud}(\mathfrak{g})_{\mathrm{As}}$.

To get an idea of how to find dialgebras let us recall that if $\mathfrak{g}$ is a Lie algebra, M is a left $\mathfrak{g}$-module and $f\colon M\to\mathfrak{g}$ is a $\mathfrak{g}$-homomorphism to the adjoint representation of $\mathfrak{g}$, then the bracket $[m,n]:=[f(m),n]$ defines a Leibniz algebra structure on M and conversely any Leibniz algebra can be obtained in this way. Quite similarly, if A is an associative algebra, M is an A-A-bimodule and $f\colon M\to A$ is a homomorphism of A-A-bimodules, then $m\dashv n:=mf(n)$ and $m\vdash n:=f(m)n$ gives a dialgebra structure on M and one can easily observe that any dialgebra can be obtained in this way.

The author constructs the corresponding homology theory for dialgebras. For a dialgebra D the chain complex $CY_*(D)$ is constructed in such a way that $CY_n(D)$ is the direct sum of c_n copies of $D^{\otimes n}$, where $c_n=(2n)!/n!(n+1)!$ is the Catalan number. It is well known that c_n computes the number of elements of Y_n, where Y_n is the set of planar binary trees with $(n+1)$ leaves and trees are involved in the definition of the boundary map of the complex $CY_*(D)$ as follows. For any i, $0\le i\le n$, there is a map $d_i\colon Y_n\to Y_{n-1}$ which assigns to a tree y the tree d_iy obtained from y by deleting the i-th leaf. For any i, $0\le i\le n-1$, there is a map $\circ_i\colon Y_n\to\{\dashv,\vdash\}$ which assigns $\circ_i^y=\dashv$ [resp. $\circ_i^y=\vdash$] if the i-th leaf points from the vertex to the left [resp. right]. Now one defines the boundary map

$$d\colon CY_n(D)=K[Y_n]\otimes D^{\otimes n}\to K[Y_{n-1}]\otimes D^{\otimes n-1}=CY_{n-1}(D)$$

by

$$d(y;a_1,\dots,a_n)=\sum_{i=1}^{n-1}(-1)^i(d_i(y);a_1,\dots,a_{i-1},a_i\circ_i^y a_{i+1},\dots,a_n).$$

The homology of the complex $CY_*(D)$ is called dialgebra homology. As was proved by the author dialgebra homology vanishes on free dialgebras. Actually the author proves that the operad corresponding to dialgebras is Koszul in the sense of V. Ginzburg and M. M. Kapranov [Duke Math. J. **76** (1994), no. 1, 203–272; MR1301191], therefore it has a dual operad, which corresponds to so-called dendriform algebras. These algebras have two operations $\prec$ and $\succ$ satisfying the following axioms:

$$(a\prec b)\prec c=a\prec(b\prec c)+a\prec(b\succ c),$$

$$(a\succ b)\prec c=a\succ(b\prec c),$$

$$(a\prec b)\succ c+(a\succ b)\succ c=a\succ(b\succ c).$$

One observes that for a dendriform algebra E the product $x*y:=x\prec y+x\succ y$ is associative. It is interesting to note that the classical shuffle algebra structure on $T(V)=\bigoplus_{n\geq 1}V^{\otimes n}$ is indeed a dendriform algebra. The author also constructs the chain complex computing the homology of dendriform algebras.

{For the collection containing this paper see MR1864390}

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